Lecture 9: Time-Domain Analysis of Discrete-Time Systems

Dr.-Ing. Sudchai Boonto

Department of Control System and Instrumentation Engineering
King Mongkut’s University of Technology Thonburi
Thailand
Outline

- Discrete-Time System Equations
- $E$ operator
- Response of Linear Discrete-Time Systems
- Useful Signal Operations
- Response of Linear Discrete-Time Systems
- Zero-Input and Zero-State Response
- Convolution Sum
- Graphical Procedure for the Convolution Sum
- Classical Solution of Linear Difference Equations
Discrete-Time System Equations

Advanced operator form:

\[ y[k + n] + a_{n-1}y[k + n - 1] + \cdots + a_1y[k + 1] + a_0y[k] = \\
\quad b_m f[k + m] + b_{m-1}f[k + m - 1] + \cdots + b_1f[k + 1] + b_0f[k] \]

The left-hand side of this form consists of the output at instants \( k + n \), \( k + n - 1 \), \( k + n - 2 \), and so on. The right-hand side of the equation consists of the input at instants \( k + m \), \( k + m - 1 \), \( k + m - 2 \), and so on.

The condition that the above system is causal is \( m \leq n \). For a general causal case, \( m = n \), the above system can be expressed as

\[ y[k + n] + a_{n-1}y[k + n - 1] + \cdots + a_1y[k + 1] + a_0y[k] = \\
\quad b_nf[k + n] + b_{n-1}f[k + n - 1] + \cdots + b_1f[k + 1] + b_0f[k] \]
Delay operator form: In case $m = n$, we can replace $k$ by $k - n$ throughout the equation. Such replacement yields the delay operator form.

\[
y[k] + a_{n-1}y[k - 1] + \cdots + a_1y[k - n + 1] + a_0y[k - n] = \\
b_nf[k] + b_{n-1}f[k - 1] + \cdots + b_1f[k - n + 1] + b_0f[k - n]
\]
Iterative Solution of Difference Equations

From the delay operator form

\[ y[k] + a_{n-1}y[k - 1] + \cdots + a_1y[k - n + 1] + a_0y[k - n] = \]
\[ b_n f[k] + b_{n-1}f[k - 1] + \cdots + b_1f[k - n + 1] + b_0f[k - n] \]

It can be expressed as

\[ y[k] = -a_{n-1}y[k - 1] - a_{n-2}y[k - 2] - \cdots - a_0y[k - n] \]
\[ + b_n f[k] + b_{n-1}f[k - 1] + \cdots + b_0f[k - n] \]

There are the past \( n \) values of the output: \( y[k - 1], y[k - 2], \ldots, y[k - n] \), the past \( n \) values of the input: \( f[k - 1], f[k - 2], \ldots, f[k - n] \), and the present value of the input \( f[k] \).
Initial Conditions and Iterative Solution of Difference Equations

If the input is causal, the \( f[-1] = f[-2] = \ldots = f[-n] = 0 \), and we need only \( n \) initial conditions \( y[-1], y[-2], \ldots, y[-n] \). This result allows us to compute iteratively or recursively the output \( y[0], y[1], y[2], y[3], \ldots \), and so on. For instance,

- to find \( y[0] \) we set \( k = 0 \).
- the left-hand side is \( y[0] \), and the right-hand side contains terms \( y[-1], y[-2], \ldots, y[-n] \) and the input \( f[0], f[-1], f[-2], \ldots, f[-n] \).
- Therefore, we must know the \( n \) initial conditions \( y[-1], y[-2], \ldots, y[-n] \) to find \( y[0], y[1], y[2], \ldots \) and so on.
Iterative Solution of Difference Equations

Examples

Solve iteratively

\[ y[k] - 0.5y[k - 1] = f[k] \]

with initial condition \( y[-1] = 16 \) and causal input \( f[k] = k^2 \).

**Solution:** Rewritten the equation in the delay operator form and move all past outputs to the left:

\[ y[k] = 0.5y[k - 1] + f[k] \]

We obtain

\[
\begin{align*}
y[0] &= 0.5y[-1] + f[0] = 0.5(16) + 0 = 8 \\
y[1] &= 0.5y[0] + f[1] = 0.5(8) + (1)^2 = 5 \\
y[2] &= 0.5y[1] + f[2] = 0.5(5) + (2)^2 = 6.5 \\
\end{align*}
\]
Iterative Solution of Difference Equations

Examples cont.

Figure: Iterative solution of a difference
Iterative Solution of Difference Equations

Examples cont.

Solve iteratively

\[ y[k + 2] - y[k + 1] + 0.24y[k] = f[k + 2] - 2f[k + 1] \]

with initial conditions \( y[-1] = 2, \ y[-2] = 1 \) and a causal input \( f[k] = k \).

**Solution:** Rewritten the equation in the delay operator form and move all past outputs to the left:

\[ y[k] = y[k - 1] - 0.24y[k - 2] + f[k] - 2f[k - 1] \]

We obtain

\[
\begin{align*}
y[0] &= y[-1] - 0.24y[-2] + f[0] - 2f[-1] = 2 - 0.24(1) + 0 - 0 = 1.76 \\
y[1] &= y[0] - 0.24y[-1] + f[1] - 2f[0] = 1.76 - 0.24(2) + 1 - 0 = 2.28 \\
\end{align*}
\]
In continuous-time system we used the operator $D$ to denote the operation of differentiation. For discrete-time systems we use the operator $E$ to denote the operation for advancing the sequence by one time unit. Thus

$$Ef[k] = f[k + 1]$$
$$E^2f[k] = f[k + 2]$$
$$\vdots$$
$$E^n f[k] = f[k + n]$$

For example

$$y[k + 1] - ay[k] = f[k + 1]$$
$$Ey[k] - ay[k] = Ef[k]$$
$$(E - a)y[k] = Ef[k]$$
For the second-order difference equation

\[ y[k + 2] + \frac{1}{4} y[k + 1] + \frac{1}{16} y[k] = f[k + 2] \]

\[ \left( E^2 + \frac{1}{4} E + \frac{1}{16} \right) y[k] = E^2 f[k] \]

A general \( n \)th-order difference equation \((n = m)\) can be expressed as

\[ (E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0) y[k] = \]

\[ (b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0) f[k] \]

\[ Q[E]y[k] = P[E]f[k] \]

where

\[ Q[E] = E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0 \]

\[ P[E] = b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0 \]
Response of Linear Discrete-Time Systems

System response to Internal Conditions: The Zero-Input Response

Similar to the continuous-time case,

Total response = zero-input response + zero-state response

The zero-input response $y_0[k]$ is the solution of the system with $f[k] = 0$; that is,

$$Q[E]y_0[k] = 0$$

or

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0) y_0[k] = 0$$

$$y_0[k + n] + a_{n-1}y_0[k + n - 1] + \cdots + a_1y_0[k + 1] + a_0y_0[k] = 0$$
The equation states that a linear combination of $y_0[k]$ and advanced $y_0[k]$ is zero not for some values of $k$ but for all $k$. Such situation is possible if and only if $y_0[k]$ and advanced $y_0[k]$ have the same form. This is true only for an exponential function $\gamma^k$. Since

$$\gamma^{k+m} = \gamma^m \gamma^k$$

Therefore, if $y_0[k] = c\gamma^k$ we have

$$Ey_0[k] = y_0[k + 1] = c\gamma^{k+1} = c\gamma \gamma^k$$

$$E^2y_0[k] = y_0[k + 2] = c\gamma^{k+2} = c\gamma^2 \gamma^k$$

$$\vdots$$

$$E^ny_0[k] = y_0[k + n] = c\gamma^{k+n} = c\gamma^n \gamma^k$$
Substitution of these results to the system equation yields

$$c \left( \gamma^n + a_{n-1} \gamma^{n-1} + \cdots + a_1 \gamma + a_0 \right) \gamma^k = 0$$

For a nontrivial solution of this equation

$$\left( \gamma^n + a_{n-1} \gamma^{n-1} + \cdots + a_1 \gamma + a_0 \right) = 0 \text{ or } Q[\gamma] = 0$$

$Q[\gamma]$ is an $n$th-order polynomial and can be expressed in the factorized form (assuming all distinct roots):

$$\left( \gamma - \gamma_1 \right) \left( \gamma - \gamma_2 \right) \cdots \left( \gamma - \gamma_n \right) = 0$$

Clearly, $\gamma$ has $n$ solutions $\gamma_1, \gamma_2, \cdots, \gamma_n$ and, the system has $n$ solutions $c_1 \gamma_1^k, c_2 \gamma_2^k, \cdots, c_n \gamma_n^k$. 

The zero-input response is

\[ y_0[k] = c_1 \gamma_1^k + c_2 \gamma_2^k + \cdots + c_n \gamma_n^k \]

where \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are the roots of the polynomial.

- \( Q[\gamma] \) is called the **characteristic polynomial** of the system.
- \( Q[\gamma] = 0 \) is the **characteristic equation** of the system.
- \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are called **characteristic roots** or **characteristic values** (also **eignevalues**) of the system.
- The exponentials \( \gamma_i^k (i = 1, 2, \ldots, n) \) are **characteristic modes** or **natural modes** of the system.
Repeated Roots:

If two or more roots are repeated, the form of the characteristic modes is modified. Similar to the continuous-time case, if a root $\gamma$ repeats $r$ times, the characteristic modes corresponding to this root are $\gamma^k$, $k\gamma^k$, $k^2\gamma^k$, \ldots, $k^{r-1}\gamma^k$.

If the characteristic equation of a system is

$$Q[\gamma] = (\gamma - \gamma_1)^r(\gamma - \gamma_{r+1})(\gamma - \gamma_{r+2})\cdots(\gamma - \gamma_n)$$

the zero-input response of the system is

$$y_0[k] = (c_1 + c_2k + c_3k^2 + \cdots + c_rk^{r-1})\gamma_1^k$$
$$+ c_{r+1}\gamma_{r+1}^k + c_{r+2}\gamma_{r+2}^k + \cdots + c_n\gamma_n^k$$
Complex Roots:
As in the case of continuous-time systems, the complex roots of a
discrete-time system must occur in pairs of conjugates so that the
system equation coefficients are real. Like the case of continuous-time
systems, we can eliminate dealing with complex numbers by using the
real form of the solution.

• First express the complex conjugate roots $\gamma$ and $\gamma^*$ in polar form.

$$\gamma = |\gamma|e^{j\beta} \quad \text{and} \quad \gamma^* = |\gamma|e^{-j\beta}$$

• the zero-input response is given by

$$y_0[k] = C_1\gamma^k + C_2(\gamma^*)^k$$
$$= C_1|\gamma|^k e^{j\beta k} + C_2|\gamma|^k e^{-j\beta k}$$
For a real system, $C_1$ and $C_2$ must be conjugates so that $y_0[k]$ is a real function of $k$. Let

$$C_1 = \frac{C}{2} e^{j\theta} \quad \text{and} \quad C_2 = \frac{C}{2} e^{-j\theta}$$

$$y_0[k] = \frac{C}{2} |\gamma|^k \left[ e^{j(\beta k + \theta)} + e^{-j(\beta k + \theta)} \right]$$

$$= C|\gamma|^k \cos(\beta k + \theta)$$

where $C$ and $\theta$ are arbitrary constants determined from the auxiliary conditions.
For an LTID system described by the difference equation

\[ y[k + 2] - 0.6y[k + 1] - 0.16y[k] = 5f[k + 2] \]

Find the zero-input response \( y_0[k] \) of the system if the initial conditions are \( y[-1] = 0 \) and \( y[-2] = \frac{25}{4} \).

The system equation in \( E \) operator form is

\[ (E^2 - 0.6E - 0.16)y[k] = 5E^2 f[k] \]

The characteristic equation is

\[ \gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma + 0.8) = 0 \]

The zero-input response is

\[ y_0[k] = C_1(-0.2)^k + C_2(0.8)^k \]
Substitute $y_0[-1] = 0$ and $y_0[-2] = \frac{25}{4}$ we obtain

\[-5C_1 + \frac{5}{4}C_2 = 0\]
\[25C_1 + \frac{25}{16}C_2 = \frac{25}{4}\]

and $C_1 = \frac{1}{5}$ and $C_2 = \frac{4}{5}$. Therefore

\[y_0[k] = \frac{1}{5}(-0.2)^k + \frac{4}{5}(0.8)^k, \quad k \geq 0.\]
A system specified by the equation

\[(E^2 + 6E + 9)y[k] = (2E^2 + 6E)f[k]\]

determine \(y_0[k]\), the zero-input response, if the initial condition are \(y_0[-1] = -\frac{1}{3}\) and \(y_0[-2] = -\frac{2}{9}\).

The characteristic equation is

\[\gamma^2 + 6\gamma + 9 = (\gamma + 3)^2 = 0\]

and we have a repeated characteristic root at \(\gamma = -3\). Hence, the zero-input response is

\[y_0[k] = (C_1 + C_2 k)(-3)^k\]

From the initial conditions we have

\[C_1 - C_2 = 1\]
\[C_1 - 2C_2 = -2\]

and \(C_1 = 4\) , \(C_2 = 3\) . Finally, we have \(y_0[k] = (4 + 3k)(-3)^k\) , \(k \geq 0\).
Find the zero-input response of an LTID system described by the equation

\[(E^2 - 1.56E + 0.81)y[k] = (E + 3)f[k]\]

when the initial conditions are \(y_0[-1] = 2\) and \(y_0[-2] = 1\).

The characteristic equation is

\[(\gamma^2 - 1.56\gamma + 0.81) = (\gamma - 0.78 - j0.45)(\gamma - 0.78 + j0.45) = 0.\]

The characteristic roots are \(0.78 \pm j0.45\); that is, \(0.9e^{\pm j \frac{\pi}{6}}\). Thus, \(|\gamma| = 0.9\) and \(\beta = \frac{\pi}{6}\), and the zero-input response is given by

\[y_0[k] = C(0.9)^k \cos(\frac{\pi}{6}k + \theta).\]

Substituting the initial conditions \(y_0[-1] = 2\) and \(y_0[-2] = 1\), we obtain

\[\frac{C}{0.9} \cos \left( -\frac{\pi}{6} + \theta \right) = \frac{C}{0.9} \left[ \cos(-\frac{\pi}{6}) \cos \theta - \sin(-\frac{\pi}{6}) \sin \theta \right] = \frac{C}{0.9} \left[ \frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta \right] = 2\]
and

\[
\frac{C}{(0.9)^2} \cos \left( -\frac{\pi}{3} + \theta \right) = \frac{C}{0.81} \left[ \cos(-\frac{\pi}{3}) \cos \theta - \sin(-\frac{\pi}{3}) \sin \theta \right] = \frac{C}{0.81} \left[ \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \right] = 1
\]

or

\[
\frac{\sqrt{3}}{1.8} C \cos \theta + \frac{1}{1.8} C \sin \theta = 2
\]

\[
\frac{1}{1.62} C \cos \theta + \frac{\sqrt{3}}{1.62} C \sin \theta = 1.
\]

We have \( C \cos \theta = 2.308 \) and \( C \sin \theta = -0.397 \). Then

\[
\theta = \tan^{-1} \left( \frac{-0.397}{2.308} \right) = -0.17 \text{ rad}
\]

Substituting \( \theta = -0.17 \text{ radian} \) in \( C \cos \theta = 2.308 \) yields \( C = 2.34 \) and

\[
y_0[k] = 2.34(0.9)^k \cos \left( \frac{\pi}{6} k - 0.17 \right), \quad k \geq 0
\]
Consider an $n$th-order system specified by the equation

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0) y[k] = (b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0) f[k]$$

or

$$Q[E]y[k] = P[E]f[k]$$

The unit impulse response $h[k]$ is the solution of this equation for the input $\delta[k]$ with all the initial conditions zero; that is

$$Q[E]h[k] = P[E]\delta[k]$$

subject to initial conditions

$$h[-1] = h[-2] = \cdots = h[-n] = 0$$
The Unit Impulse Response $h[k]$

$h[k]$ is the system response to the input $\delta[k]$, which is zero for $k > 0$. We know that when the input is zero, only the characteristic modes can be sustained by the system. Therefore, $h[k]$ must be made up of characteristic modes for $k > 0$. At $k = 0$, it may have some nonzero value, and $h[k]$ can be expressed as

$$h[k] = \frac{b_0}{a_0} \delta[k] + y_n[k]u[k].$$

The $n$ unknown coefficients in $y_n[k]$ can be determined from a knowledge of $n$ values of $h[k]$. It is a straightforward task to determine values of $h[k]$ iteratively.
For a discrete-time system specified above, we have

\[ h[k] = A_0 \delta[k] + y_n[k] u[k] \]

Then

\[ Q[E](A_0 \delta[k] + y_n[k] u[k]) = P[E] \delta[k] \]

because \( y_n[k] u[k] \) is a sum of characteristic modes

\[ Q[E](y_n[k] u[k]) = 0, \quad k \geq 0 \]

The above equation reduces to

\[ A_0 Q[E] \delta[k] = P[E] \delta[k], \quad k \geq 0 \]
The Unit Impulse Response $h[k]$

The Closed-Form Solution of $h[k]$ Deviation cont.

or

$$A_0 \left( E^n + a_{n-1} E^{n-1} + \cdots + a_1 E + a_0 \right) \delta[k]$$

$$= \left( b_n E^n + b_{n-1} E^{n-1} + \cdots + b_1 E + b_0 \right) \delta[k]$$

$$A_0 (\delta[k + n] + a_{n-1} \delta[k + n - 1] + \cdots + a_1 \delta[k + 1] + a_0 \delta[k])$$

$$= b_n \delta[k + n] + b_{n-1} \delta[k + n - 1] + \cdots + b_1 \delta[k + 1] + b_0 \delta[k]$$

If we set $k = 0$ in the equation and recognize that $\delta[0] = 1$ and $\delta[m] = 0$ when $m \neq 0$, all but the last terms vanish on both sides, yielding

$$A_0 a_0 = b_0 \quad \text{and} \quad A_0 = \frac{b_0}{a_0}$$

Note: for the special case $a_0 = 0$ see the reference.
The Unit Impulse Response $h[k]$

Example

Determine the unit impulse response $h[k]$ for a system specified by the equation

$$y[k] - 0.6y[k - 1] - 0.16y[k - 2] = 5f[k]$$

This equation can be expressed in the advance operator form as

$$y[k + 2] - 0.6y[k + 1] - 0.16y[k] = 5f[k + 2]$$

or

$$(E^2 - 0.6E - 0.16) y[k] = 5E^2 f[k]$$

The characteristic equation is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8) = 0.$$ 

Therefore

$$y_n[k] = C_1(-0.2)^k + C_2(0.8)^k$$
The Unit Impulse Response $h[k]$

Example cont.

From the system we have $a_0 = -0.16$ and $b_0 = 0$. Therefore

$$h[k] = \frac{0}{-0.16} \delta[k] + \left[ C_1(-0.2)^k + C_2(0.8)^k \right] u[k] = \left[ C_1(-0.2)^k + C_2(0.8)^k \right] u[k]$$

To determine $C_1$ and $C_2$, we need to find two values of $h[k]$ iteratively. To do this, we must let the input $f[k] = \delta[k]$ and the output $y[k] = h[k]$ in the system equation. The resulting equation is

$$h[k] - 0.6h[k - 1] - 0.16h[k - 2] = 5\delta[k]$$

subject to zero initial state; that is, $h[-1] = h[-2] = 0$.

Setting $k = 0$ in this equation yields

$$h[0] - 0.6(0) - 0.16(0) = 5(1) \implies h[0] = 5$$

Next, setting $k = 1$ and using $h[0] = 5$, we obtain

$$h[1] - 0.6(5) - 0.16(0) = 5(0) \implies h[1] = 3$$
The Unit Impulse Response $h[k]$

Example cont.

Then we have

$$h[0] = C_1(-0.2)^0 + C_2(0.8)^0 = C_1 + C_2 = 5$$

$$h[1] = C_1(-0.2)^1 + C_2(0.8)^1 = -0.2C_1 + 0.8C_2 = 3$$

and $C_1 = 1$, $C_2 = 4$. Therefore

$$h[k] = \left[ (-0.2)^k + 4(0.8)^k \right] u[k]$$
The zero-state response $y[k]$ is the system response to an input $f[k]$ when the system is in zero state. Like in the continuous-time case an arbitrary input $f[k]$ can be expressed as a sum of impulse components.
The previous page shows how a signal $f[k]$ can be expressed as a sum of impulse components. The component of $f[k]$ at $k = m$ is $f[m] \delta[k - m]$, and $f[k]$ is the sum of all these components summed from $m = -\infty$ to $\infty$. Therefore

$$f[k] = f[0] \delta[k] + f[1] \delta[k - 1] + f[2] \delta[k - 2] + \cdots$$
$$+ f[-1] \delta[k + 1] + f[-2] \delta[k + 2] + \cdots$$
$$= \sum_{m=-\infty}^{\infty} f[m] \delta[k - m]$$

If we knew the system response to impulse $\delta[k]$, the system response to any arbitrary input could be obtained by summing the system response to various impulse components.
System Response to External Input

The Zero-State Response cont.

If

\[ \delta[k] \implies h[k] \]

then

\[ \delta[k - m] \implies h[k - m] \]

\[ f[m] \delta[k - m] \implies f[m] h[k - m] \]

\[ \sum_{m=-\infty}^{\infty} f[m] \delta[k - m] \implies \sum_{m=-\infty}^{\infty} f[m] h[k - m] \]

\[ \underbrace{f[k]}_{f[k]} \quad \underbrace{y[k]}_{y[k]} \]
We have the response \( y[k] \) to input \( f[k] \) as

\[
y[k] = \sum_{m=-\infty}^{\infty} f[m]h[k - m].
\]

This summation on the right-hand side is known as the convolution sum of \( f[k] \) and \( h[k] \), and is represented symbolically by \( f[k] * h[k] \)

\[
f[k] * h[k] = \sum_{m=-\infty}^{\infty} f[m]h[k - m]
\]
Properties of the Convolution Sum

The Commutative Property

The Commutative Property

\[ f_1[k] * f_2[k] = f_2[k] * f_1[k] \]

This can be proved as follow:

\[
\begin{align*}
  f_1[k] * f_2[k] &= \sum_{m=-\infty}^{\infty} f_1[m] f_2[k - m] \\
  &= - \sum_{w=\infty}^{-\infty} f_1[w - k] f_2[w], \quad w = k - m \\
  &= \sum_{w=-\infty}^{\infty} f_2[w] f_1[w - k] \\
  &= f_2[k] * f_1[k]
\end{align*}
\]
Properties of the Convolution Sum
The Distributive Property

\[ f_1[k] \ast (f_2[k] + f_3[k]) = f_1[k] \ast f_2[k] + f_1[k] \ast f_3[k] \]

The proof is as follow:

\[
\begin{align*}
  f_1[k] \ast (f_2[k] + f_3[k]) &= \sum_{m=-\infty}^{\infty} f_1[m] (f_2[k - m] + f_3[k - m]) \\
  &= \sum_{m=-\infty}^{\infty} f_1[m] f_2[k - m] + \sum_{m=-\infty}^{\infty} f_1[m] f_3[k - m] \\
  &= f_1[k] \ast f_2[k] + f_1[k] \ast f_3[k]
\end{align*}
\]
The Associative Property

\[ f_1[k] * (f_2[k] * f_3[k]) = (f_1[k] * f_2[k]) * f_3[k] \]

The proof is as follow:

\[
\begin{align*}
 f_1[k] * (f_2[k] * f_3[k]) &= \sum_{m_1=-\infty}^{\infty} f_1[m_1] (f_2[k - m_1] * f_3[k - m_1]) \\
 &= \sum_{m_1=-\infty}^{\infty} f_1[m_1] \sum_{m_2=-\infty}^{\infty} f_2[m_2] f_3[k - m_1 - m_2] \\
 &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_1[\lambda - m_2] f_2[m_2] f_3[k - \lambda]
\end{align*}
\]

, where \( \lambda = m_1 + m_2 \).
Properties of the Convolution Sum

The Associative Property cont.

Then we have

\[ f_1[k] \ast (f_2[k] \ast f_3[k]) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_1[\lambda - m_2]f_2[m_2]f_3[k - \lambda] \]

\[ = (f_1[k] \ast f_2[k]) \ast f_3[k] \]

The Convolution with an Impulse

\[ f[k] \ast \delta[k] = \sum_{m=-\infty}^{\infty} f[m]\delta[k - m] \]

Since \( \delta[k - m] = 1 \), if \( k - m = 0 \) or \( m = k \), then

\[ f[k] \ast \delta[k] = f[k]. \]
The shifting Property

If

\[ f_1[k] * f_2[k] = c[k] \]

then

\[ f_1[k] * f_2[k - n] = f_1[k] * f_2[k] * \delta[k - n] \]
\[ = c[k] * \delta[k - n] = c[k - n] \]
\[ f_1[k - n] * f_2[k] = f_1[k] * \delta[k - n] * f_2[k] \]
\[ = f_1[k] * f_2[k] * \delta[k - n] \]
\[ = c[k] * \delta[k - n] = c[k - n] \]
\[ f_1[k - n] * f_2[k - l] = f_1[k] * \delta[k - n] * f_2[k] * \delta[k - l] \]
\[ = c[k] * \delta[k - n] * \delta[k - l] = c[k - n - l] \]
Properties of the Convolution Sum

The shifting Property

The Width Property
If \( f_1[k] \) and \( f_2[k] \) have lengths of \( m \) and \( n \) elements respectively, then the length of \( c[k] \) is \( m + n - 1 \) elements.
Causality and Zero-State Response

- We assumed the system to be linear and time-invariant.
- In practice, almost all of the input signals are causal, and a majority of the system are also causal.
- If the input $f[k]$ is causal, then $f[m] = 0$ for $m < 0$.
- Similarly, if the system is causal, then $h[x] = 0$ for negative $x$, so that $h[k - m] = 0$ when $m > k$.
- Therefore, if $f[k]$ and $h[k]$ are both causal, the product $f[m]h[k - m] = 0$ for $m < 0$ and for $m > k$, and it is nonzero only for the range $0 \leq m \leq k$. Therefore, the convolution sum is reduced to

$$y[k] = \sum_{m=0}^{k} f[k]h[k - m]$$
Convolution Sum

Analytical Method Example

Determine $c[k] = f[k] \ast g[k]$ for

$$f[k] = (0.8)^k u[k] \text{ and } g[k] = (0.3)^k u[k]$$

we have

$$c[k] = \sum_{m=0}^{k} f[m] g[k-m]$$

since both signals are causal.

$$c[k] = \begin{cases} \sum_{m=0}^{k} (0.8)^m (0.3)^{k-m} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$c[k] = (0.3)^k \sum_{m=0}^{k} \left( \frac{0.8}{0.3} \right)^m u[k] = (0.3)^k \frac{(0.8)^{k+1} - (0.3)^{k+1}}{(0.3)^k (0.8 - 0.3)} u[k]$$

$$= 2 \left[ (0.8)^{k+1} - (0.3)^{k+1} \right] u[k]$$
Zero-State Response
Analytical Method Example

Find the zero-state response \( y[k] \) of an LTID system described by the equation

\[
y[k + 2] - 0.6y[k + 1] - 0.16y[k] = 5f[k + 2]
\]

if the input \( f[k] = 4^{-k}u[k] \) and \( h[k] = [(-0.2)^k + 4(0.8)^k] u[k] \).

We have

\[
y_i[k] = f[k] * h[k]
\]

\[
= (4)^{-k}u[k] * [(-0.2)^k u[k] + 4(0.8)^k u[k]]
\]

\[
= (4)^{-k}u[k] * (-0.2)^k u[k] + (4)^{-k}u[k] * 4(0.8)^k u[k]
\]

\[
= (0.25)^k u[k] * (-0.2)^k u[k] + 4(0.25)^k u[k] * (0.8)^k u[k]
\]

Using Pair 4 from the convolution sum table:

\[
y[k] = \left[ \frac{(0.25)^{k+1} - (-0.2)^{k+1}}{0.25 - (-0.2)} + 4 \frac{(0.25)^{k+1} - (0.8)^{k+1}}{0.25 - 0.8} \right] u[k]
\]
Zero-State Response
Analytical Method Example cont.

\[ y[k] = \left(2.22 \left((0.25)^{k+1} - (-0.2)^{k+1}\right) - 7.27 \left((0.25)^{k+1} - (0.8)^{k+1}\right)\right) u[k] \]

\[ = \left[-5.05(0.25)^{k+1} - 2.22(-0.2)^{k+1} + 7.27(0.8)^{k+1}\right] u[k] \]

Recognizing that

\[ \gamma^{k+1} = \gamma(\gamma)^k \]

We can express \( y[k] \) as

\[ y[k] = \left[-1.26(0.25)^k + 0.444(-0.2)^k + 5.81(0.8)^k\right] u[k] \]

\[ = \left[-1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k\right] u[k] \]
Graphical Procedure for the Convolution Sum

The convolution sum of causal signals $f[k]$ and $g[k]$ is given by

$$c[k] = \sum_{m=0}^{k} f[k]g[k - m]$$

- Invert $g[m]$ about the vertical axis ($m = 0$) to obtain $g[-m]$.
- Time shift $g[-m]$ by $k$ units to obtain $g[k - m]$. For $k > 0$, the shift is to the right (delay); for $k < 0$, the shift is to the left (advance).
- Next we multiply $f[m]$ and $g[k - m]$ and add all the products to obtain $c[k]$. The procedure is repeated to each value of $k$ over the range $-\infty$ to $\infty$. 
Find $c[k] = f[k] * g[k]$, where $f[k]$ and $g[k]$ are depicted in the Figures.
The two functions $f[m]$ and $g[k - m]$ overlap over the interval $0 \leq m \leq k$. 
Graphical Procedure for the Convolution Sum

Example

Therefore

\[ c[k] = \sum_{m=0}^{k} f[m]g[k-m] \]

\[ = \sum_{m=0}^{k} (0.8)^m (0.3)^{k-m} \]

\[ = (0.3)^k \sum_{m=0}^{k} \left( \frac{0.8}{0.3} \right)^m \]

\[ = 2 \left[ (0.8)^{k+1} - (0.3)^{k+1} \right], \quad k \geq 0 \]

For \( k < 0 \), there is no overlap between \( f[m] \) and \( g[k-m] \), so that \( c[k] = 0 \) \( k < 0 \) and

\[ c[k] = 2 \left[ (0.8)^{k+1} - (0.3)^{k+1} \right] u[k]. \]
Using the sliding tape method, convolve the two sequences $f[k]$ and $g[k]$.

- write the sequences $f[k]$ and $g[k]$ in the slots of two tapes
- leave the $f$ tape stationary (to correspond to $f[m]$). The $g[-m]$ tape is obtained by time inverting the $g[m]$ tape
- shift the inverted tape by $k$ slots, multiply values on two tapes in adjacent slots, and add all the products to find $c[k]$. 

Graphical Procedure for the Convolution Sum

Sliding Tape Method

Using the sliding tape method, convolve the two sequences $f[k]$ and $g[k]$. 

- write the sequences $f[k]$ and $g[k]$ in the slots of two tapes 
- leave the $f$ tape stationary (to correspond to $f[m]$). The $g[-m]$ tape is obtained by time inverting the $g[m]$ tape 
- shift the inverted tape by $k$ slots, multiply values on two tapes in adjacent slots, and add all the products to find $c[k]$. 

For the case of $k = 0$,

$$c[0] = 0 \times 1 = 0$$

For $k = 1$

$$c[1] = (0 \times 1) + (1 \times 1) = 1$$

Similarly,

$$c[2] = (0 \times 1) + (1 \times 1) + (2 \times 1) = 3$$
$$c[3] = (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) = 6$$
$$c[4] = (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) = 10$$
$$c[5] = (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) + (5 \times 1) = 15$$
$$c[6] = (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) + (5 \times 1) = 15$$

...
Graphical Procedure for the Convolution Sum

Sliding Tape Method cont.

Lecture 9: Time-Domain Analysis of Discrete-Time Systems
Graphical Procedure for the Convolution Sum

Sliding Tape Method cont.

\[
\begin{align*}
\text{Graphical Representation:} & \quad f[m] = [0, 1, 2, 3, 4, 5] \\
& \quad c[3] = 6 \\
& \quad f[m] = [0, 1, 2, 3, 4, 5] \\
& \quad c[4] = 10 \\
& \quad f[m] = [0, 1, 2, 3, 4, 5] \\
& \quad c[5] = 15 \\
& \quad f[m] = [0, 1, 2, 3, 4, 5] \\
& \quad c[6] = 15
\end{align*}
\]
The total response of an LTID system can be expressed as a sum of the zero-input and zero-state components:

\[
\begin{align*}
\text{Total response } y[k] &= \sum_{j=1}^{n} c_j \gamma_j^k + f[k] * h[k] \\
&= \text{Zero-input component} + \text{Zero-state component}
\end{align*}
\]

From the previous example, the system described by the equation

\[
y[k + 2] - 0.6y[k + 1] - 0.16y[k] = 5f[k + 2]
\]

with initial conditions \( y[-1] = 0, y[-2] = \frac{25}{4} \) and input \( f[k] = (4)^{-k}u[k] \). We have

\[
y[k] = 0.2(-0.2)^k + 0.8(0.8)^k - 1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k
\]

Zero-input component

Zero-state component
Classical solution of Linear Difference Equations

If $y_n[k]$ and $y_\phi[k]$ denote the natural and the forced response respectively, the total response is given by

$$\text{Total response } y[k] = y_n[k] + y_\phi[k]$$

Because $y_n[k] + y_\phi[k]$ is a solution of the system, we have

$$Q[E](y_n[k] + y_\phi[k]) = P[E]f[k]$$

$y_n[k]$ is made up of characteristic modes,

$$Q[E]y_n[k] = 0$$

Substitution of this equation yields

$$Q[E]y_\phi[k] = P[E]f[k]$$
By definition, the forced response contains only nonmode terms and the list of the inputs and the corresponding forms of the forced function is show below:

<table>
<thead>
<tr>
<th>Input $f[k]$</th>
<th>Forced Response $y_\phi[k]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^k r \neq \gamma_i (i = 1, 2, \ldots, n)$</td>
<td>$c r^k$</td>
</tr>
<tr>
<td>$r^k r = \gamma_i$</td>
<td>$ck r^k$</td>
</tr>
<tr>
<td>$\cos(\beta k + \theta)$</td>
<td>$c \cos(\beta k + \phi)$</td>
</tr>
<tr>
<td>$\left(\sum_{i=0}^{m} \alpha_i k^i\right) r^k$</td>
<td>$\left(\sum_{i=0}^{m} c_i k^i\right) r^k$</td>
</tr>
</tbody>
</table>

Note: By definition $y_\phi[k]$ cannot have any characteristic mode terms.
Classical solution of Linear Difference Equations

Forced Response example

Determine the total response $y[k]$ of a system

$$(E^2 - 5E + 6)y[k] = (E - 5)f[k]$$

if the input $f[k] = (3k + 5)u[k]$ and the auxiliary conditions are $y[0] = 4, y[1] = 13$. The characteristic equation is

$$\gamma^2 - 5\gamma + 6 = (\gamma - 2)(\gamma - 3) = 0$$

Therefore, the natural response is

$$y_n[k] = B_1(2)^k + B_2(3)^k$$

To find the form of forced response $y_\phi[k]$, we use above Table, Pair 4 with $r = 1, m = 1$. This yields

$$y_\phi[k] = c_1k + c_0$$
Classical solution of Linear Difference Equations

Forced Response example cont.

Therefore

\[ y_φ[k + 1] = c_1(k + 1) + c_0 = c_1 k + c_1 + c_0 \]
\[ y_φ[k + 2] = c_1(k + 2) + c_0 = c_1 k + 2c_1 + c_0 \]

Also

\[ f[k] = 3k + 5 \]

and

\[ f[k + 1] = 3(k + 1) + 5 = 3k + 8 \]

Substitution of the above results yields

\[ c_1 k + 2c_1 + c_0 - 5(c_1 k + c_1 + c_0) + 6(c_1 k + c_0) = 3k + 8 - 5(3k + 5) \]
\[ 2c_1 k - 3c_1 + 2c_0 = -12k - 17 \]
Classical solution of Linear Difference Equations

Forced Response example cont.

Comparison of similar terms on the two sides yields

\[ 2c_1 = -12 \]
\[ -3c_1 + 2c_0 = -17 \]

and \( c_1 = -6, \ c_2 = -\frac{35}{2} \). Therefore

\[ y_\phi[k] = -6k - \frac{35}{2} \]

The total response is

\[ y[k] = y_n[k] + y_\phi[k] \]
\[ = B_1(2)^k + B_2(3)^k - 6k - \frac{35}{2}, \ k \geq 0 \]
To determine arbitrary constants $B_1$ and $B_2$ we set $k = 0$ and $1$ and substitute the initial conditions $y[0] = 4$, $y[1] = 13$ to obtain

$$B_1 + B_2 - \frac{35}{2} = 4$$
$$2B_1 + 3B_2 - \frac{47}{2} = 13$$

and $B_1 = 28$, $B_2 = -\frac{13}{2}$.

Therefore

$$y_n[k] = 28(2)^k - \frac{13}{2}(3)^k$$

and

$$y[k] = 28(2)^k - \frac{13}{2}(3)^k - 6k - \frac{35}{2}$$

$y_n[k]$ and $y_\phi[k]$