Lecture 12: Robust Stability and Robust Performance Analysis

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Feedback System with Uncertainty

Terminologies

- **Nominal stability (NS)**: Feedback system is internally stable when $\Delta = 0$.

- **Robust stability (RS)**: Feedback system is internally stable for any norm-bounded $\Delta$.

- **Nominal performance (NP)**: Feedback system is stable and satisfies certain performance for $\Delta = 0$.

- **Robust performance (RP)**: Feedback system is stable and satisfies certain performance for any norm-bounded $\Delta$. 
Uncertain Models

Model sets

\[ G_p(s) \in \{ G(s) + \Delta \mid \| \Delta \| \leq \gamma \} \]

\[ G(s) = \text{Nominal plant} \]
\[ \Delta = \text{unknown, but bounded, perturbation (i/o operator)} \]

Typically, \( \Delta \) is stable, causal and satisfies, \( \| \Delta \|_\infty \leq \gamma \).
The loop transfer function is

\[ L_p = G_P K = GK(1 + w_I \Delta_I) = L + w_I L \Delta_I, \quad |\Delta_I(j\omega)| \leq 1, \forall \omega \]

- the system is NP and \( L_p \) is stable

\[ RS \iff \text{System stable} \ \forall L_p \]

\[ \iff L_p \text{should not encircle the point } -1, \ \forall L_p \]
SISO Robust Stability

RS condition

- \(| -1 - L| = |1 + L|\) is the distance from the point -1 to the center of the disc representing \(L_p\), and \(|w_1L|\) is the radius of the disc.

\[
\text{RS} \iff |w_1L| < |1 + L|, \quad \forall \omega \iff \left| \frac{w_1L}{1 + L} \right| < 1, \quad \forall \omega
\]

\[
\iff |w_1T| < 1, \quad \forall \omega \iff \|w_1T\|_\infty < 1
\]
Consider the following nominal plant and PI-controller

\[ G(s) = \frac{3(-2s + 1)}{(5s + 1)(10s + 1)}, \quad K(s) = K_c \frac{12.7s + 1}{12.7s}, \quad w_I(s) = \frac{10s + 0.33}{(10/5.25)s + 1}, \]

\[ K_{c1} = 1.13, K_{c2} = 0.31 \]
SISO Robust Stability

$M\Delta$-Structure

Consider a transfer function of the $\Delta$ output to $\Delta$ input of the feedback system with multiplicative uncertainty. We have

$$w_I K (1 + G K)^{-1} G = w_I T = M$$

- The Nyquist stability condition then determines RS if and only if the “loop transfer function” $M\Delta$ does not encircle -1 for all $\Delta$. 
SISO Robust Stability

\( M\Delta \)-Structure

\[ RS \iff |1 + M\Delta| > 0, \quad \forall \omega, \forall |\Delta| \leq 1 \]

The condition is most easily violated (the worst case) when \( \Delta \) is selected at each frequency such that \( |\Delta| = 1 \) and the terms \( M\Delta \) and 1 have opposite signs (point to the opposite direction). We therefore get

\[ RS \iff 1 - |M(j\omega)| > 0, \quad \forall \omega \]
\[ \iff |M(j\omega)| < 1, \quad \forall \omega \]
\[ = \|\omega I T\| < 1 \]
SISO Robust Performance

Nominal performance

\[ |w_P(j\omega)| \]

\[ |1 + L(j\omega)| \]

\[ L(j\omega) \]

\[ NP \Leftrightarrow |w_PS| < 1 \quad \forall \omega \Leftrightarrow |w_P| < |1 + L| \quad \forall \omega \]
SISO Robust Performance

Robust performance

For robust performance we need the previous condition to be satisfied for all possible plants, that is, including the worst-case uncertainty.

$$\text{RP} \iff |w_p S_p| < 1 \quad \forall S_p, \forall \omega$$

$$\iff |w_p| < |1 + L_p| \quad \forall L_p, \forall \omega$$

This corresponds to requiring $|\hat{y}/d| < 1 \forall \Delta_I$, where we consider multiplicative uncertainty, and the set of possible loop transfer functions is

$$L_p = G_p K = L(1 + w_I \Delta_I) = L + w_I L \Delta_I$$
SISO Robust Performance

Robust performance

For RP we must require that all possible $L_p(j\omega)$ stay outside a disc of radius $|w_P(j\omega)|$ centered on -1. Since $L_p$ at each frequency stays within a disc of radius $w_IL$ centered on $L$, we see that the condition for RP is that the two discs, with radii $|w_P|$ and $|w_IL|$, do not overlap.
SISO Robust Performance

Robust performance

Since their centers are located a distance $|1 + L|$ apart, the RP-condition becomes

$$\text{RP} \iff |w_P| + |w_IL| < |1 + L|, \quad \forall \omega$$

$$\iff |w_P (1 + L)^{-1}| + |w_IL (1 + L)^{-1}| < 1, \quad \forall \omega$$

or in other words

$$\text{RP} \iff \max_{\omega} (|w_P S| + |w_I T|) < 1$$
SISO Robust Performance

Example

Consider robust performance of the SISO system in Figure, for which we have

\[ \text{RP} \iff \left| \frac{\hat{y}}{d} \right| < 1, \quad \forall \omega; \quad w_p(s) = 0.25 + \frac{0.1}{s}; \quad w_u(s) = r_u \frac{s}{s + 1} \]

1. Derive a condition for robust performance (RP).
2. For what values of \( r_u \) is it impossible to satisfy the robust performance condition?
3. Let \( r_u = 0.5 \), consider two cases for the nominal loop transfer function: 1) \( GK_1(s) = 0.5/s \) and 2) \( GK_2(s) = \frac{0.5}{s} \frac{1-s}{1+s} \). For each system, sketch the magnitudes of \( S \) and its performance bound as a function of frequency. Does each system satisfy robust performance?
SISO Robust Performance

Example

a) the requirement for RP is $|w_P S_p| < 1$, $\forall S_p, \forall \omega$, where the possible sensitivity are given by

$$S_p = \frac{1}{1 + G K + w_u \Delta_u} = \frac{S}{1 + w_u \Delta_u S}$$

The condition for RP then becomes

$$RP \iff \left| \frac{w_P S}{1 + w_u \Delta_u S} \right| < 1, \forall \Delta_u, \forall \omega$$

A simple analysis shows that the worst case corresponds to selecting $\Delta_u$ with magnitude 1 such that the term $w_u \Delta_u S$ is purely real and negative, and hence we have

$$RP \iff |w_P S| < 1 - |w_u S|, \forall \omega$$
$$\iff |w_P S| + |w_u S| < 1, \forall \omega$$
$$\iff |S(j\omega)| < \frac{1}{|w_P(j\omega)| + |w_u(j\omega)|}, \forall \omega$$
b) Since any real system is strictly proper we have $|S| = 1$ at high frequencies and therefore we must require $|w_u(j\omega)| + |w_p(j\omega)| < 1$ as $\omega \to \infty$. With the weight given, this is equivalent to $r_u + 0.25 < 1$. Therefore, we must at least require $r_u < 0.75$ for RP, so RP cannot be satisfied if $r_u \geq 0.75$. 

![Frequency Magnitude Graph](image-url)
SISO Robust Performance

Example

c) Design $S_1$ yields RP, while $S_2$ does not. This is seen by checking the RP-condition graphically as shown in Figure above; $|S_1|$ has a peak of 1 while $|S_2|$ has a peak of about 2.45.
General Control Configuration with Uncertainty

The uncertain perturbations in a block diagonal matrix,

\[
\Delta = \text{diag}\{\Delta_i\} = \begin{bmatrix}
\Delta_1 \\
\vdots \\
\Delta_i \\
\vdots 
\end{bmatrix}
\]

where each \( \Delta_i \) represents a specific source of uncertainty

\( \Delta_I = \) input uncertainty

\( \delta_i = \) parametric uncertainty where \( \delta_i \) is real.
General Control Configuration with Uncertainty

Figure: General control configuration

\[
N = \mathcal{F}_l(P, K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
\]

\[
F = \mathcal{F}_u(N, \Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}
\]

Figure: $N\Delta$-structure for robust performance analysis
To analyze robust stability of $F$, we can rearrange the system into the $M\Delta$-structure where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.
Obtaining $P, N$ and $M$

The inputs are $[u_\Delta \ w \ u]^T$ and outputs $[y_\Delta \ z \ v]^T$. By writing down the equations we get

$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_PG & W_P & W_PG \\ -G & -I & -G \end{bmatrix}, \quad P_{11} = \begin{bmatrix} 0 & 0 \\ W_PG & W_P \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} -G & -I \end{bmatrix}, \quad P_{22} = -G.$$
Obtaining $P, N$ and $M$

Find $N$ from $N = \mathcal{F}_l(P, K)$ or directly from the system we get

$$N = \begin{bmatrix} -W_IKG(I + KG)^{-1} & -W_IK(I + GK)^{-1} \\ W_PG(I + KG)^{-1} & W_P(I + GK)^{-1} \end{bmatrix}$$

The upper left block, $N_{11}$ is the transfer function from $u_\Delta$ to $y_\Delta$. This is the transfer function $M$ for $M\Delta$-structure for evaluating robust stability. Thus, we have

$$M = -W_IKG(I + KG)^{-1} = -W_IT_I$$
Robust Stability of the $M\Delta$-Structure

Consider the uncertain $N\Delta$-system for which the transfer function from $w$ to $z$ is given by

$$F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

- Suppose the system is nominally stable (with $\Delta = 0$), that is, $N$ is stable (which means that the whole of $N$, and not only $N_{22}$ must be stable).
- The only possible source of instability is the feedback term $(I - N_{11}\Delta)^{-1}$.
- The nominal stability (NS), the stability of the system is equivalent to the stability of the $M\Delta$-structure where $M = N_{11}$. 
Theorem (Determinant stability condition)

For a fixed stable $M(s)$, the $M\Delta$-structure system is internally stable for any structured $\Delta$ with $||\Delta||_{\infty} \leq 1$ if and only if

1. Nyquist plot of $\det(I - M\Delta(s))$ does not encircle the origin $\forall \Delta$ (1)
2. $\det(I - M\Delta(j\omega)) \neq 0$, $\forall \Delta$ (2)
3. $\lambda_i(M\Delta) \neq 1$, $\forall i, \forall \omega, \forall \Delta$ (3)

Proof:

- The first condition is simply the generalized Nyquist Theorem applied to a positive feedback system with a stable loop transfer function $M\Delta$. 
Robust Stability of the $M\Delta$-Structure

- (1) ⇒ (2): This is obvious since by “encirclement of the origin” we also include the origin itself.

- (2) ⇐ is proved by proving not(1) ⇒ not(2): First note that with $\Delta = 0$, $\det I - M\Delta = 1$ at all frequencies. Assume there exists a perturbation $\Delta'$ such that the image of $\det(I - M\Delta'(s))$ encircles the origin as $s$ traverses the Nyquist $D$-contour. Because the Nyquist contour and its map is closed, there then exists another perturbation in the set, $\Delta'' = \epsilon\Delta'$ with $\epsilon \in [0, 1]$, and an $\omega'$ such that $\det(I - M\Delta''(j\omega')) = 0$.

- (3) is equivalent to (2) since $\det(I - A) = \prod_i \lambda_i(I - A)$ and $\lambda_i(I - A)$ and $\lambda_i(I - A) = 1 - \lambda_i(A)$. 

Lecture 12: Robust Stability and Robust Performance Analysis
Robust Stability of the $M\Delta$-Structure

Theorem (Spectral radius condition for complex perturbations)

Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Consider the class of perturbations, $\Delta$, such that if $\Delta'$ is an allowed perturbation then so is $c\Delta'$ where $c$ is any complex scalar such that $|c| \leq 1$. Then the $M\Delta$-system is stable for all allowed perturbations if and only if

$$\rho(M\Delta(j\omega)) < 1, \quad \forall \omega, \forall \Delta$$

or equivalently

$$RS \iff \max_{\Delta} \rho(M\Delta(j\omega)) < 1, \quad \forall \omega$$
RS for Complex Unstructured Uncertainty

Theorem (RS for Unstructured Perturbations)

Assume that the nominal system $M(s)$ is stable (NS) and that the perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system is stable for all perturbations $\Delta$ satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$\bar{\sigma}(M(j\omega)) < 1, \quad \forall \omega \iff \|M\|_\infty < 1$$

Proof: We can show that

$$\det(I - M\Delta) \neq 0, \quad \forall \omega, \forall \Delta \iff \lambda_i(M\Delta) < 1, \quad \forall i, \forall \omega, \forall \Delta$$

For $\Delta$ that $\bar{\Delta} \leq 1$, we have

$$\max_{\Delta} \rho(M\Delta) = \max_{\Delta} \bar{\sigma}(M\Delta) = \max_{\Delta} \bar{\sigma}(M)\bar{\sigma}(\Delta) = \bar{\sigma}(M)$$

Then RS $\iff \bar{\sigma}(M(j\omega)) < 1, \quad \forall \omega$. 

Lecture 12: Robust Stability and Robust Performance Analysis
RS with Structured Uncertainty

- Consider the presence of structured uncertainty, where \( \Delta = \text{diag}\{\Delta_i\} \) is block diagonal. The test for robust stability is changed to

\[
\text{RS if } \bar{\sigma}(M(j\omega)) < 1, \quad \forall \omega
\]

Here we write “if” rather than “if and only if” since this condition is only sufficient for RS when \( \Delta \) has “no structure”.

- To take the advantage of the fact that \( \Delta = \text{diag}\{\Delta_i\} \) is structured to obtain an RS-condition which is tighter than the unstructured one. We can use the block-diagonal scaling matrix

\[
D = \text{diag}\{d_iI_i\}
\]

where \( d_i \) is a scalar and \( I_i \) is an identity matrix of the same dimension as the \( \Delta_i \).
RS with Structured Uncertainty

- Moreover we have $\Delta D = D\Delta$. This means the RS condition must also apply if we replace $M$ by $DMD^{-1}$ and we have

$$\text{RS if } \bar{\sigma}(DMD^{-1}) < 1, \quad \forall \omega$$
Structured Singular Value $\mu$

The structured singular value ($\mu$) is a function which provides a generalization of the singular value, $\bar{\sigma}$, and the spectral radius, $\rho$. $\mu$ can be used to get necessary and sufficient conditions for RS and RP.

Definition (Structured Singular Value)
Let $M$ be a given complex matrix and let $\Delta = \text{diag}\{\Delta_i\}$ denote a set of complex matrices matrices with $\bar{\sigma}(\Delta) \leq 1$ and with a given block-diagonal structure. The real non-negative function $\mu(M)$, called the structured singular value, is defined by

$$\mu(M) \triangleq \left( \min_{\Delta} \{ k_m | \det(I - k_m M \Delta) = 0, \quad \bar{\sigma}(\Delta) \leq 1 \} \right)^{-1}$$

If no such structured $\Delta$ exists then $\mu(M) = 0$. 
Robust Stability and Performance with Structured Uncertainty

**Theorem (RS for block-diagonal perturbations)**

Assume that the nominal system $M$ and the perturbations $\Delta$ are stable. Then the $M\Delta$-system is stable for all allowed perturbations with $\bar{\sigma}(\Delta) \leq 1$, $\forall \omega$, if and only if

$$\mu(M(j\omega)) < 1, \quad \forall \omega$$

**Theorem (RP for block-diagonal perturbations)**

Rearrange the uncertain system into the $N\Delta$-structure. Assume nominal stability such that $N$ is stable. Then

$$RS \iff \mu_{\Delta}(N(j\omega)) < 1, \quad \forall \omega.$$
μ-Synthesis

- At present there is no direct method to synthesize a μ-optimal controller. However, for complex perturbations a method known as $DK$-iteration is available.
- The method combines $\mathcal{H}_\infty$-synthesis and μ-analysis, and often yields good results.
- The idea is to find the controller that minimizes the peak value over frequency of this upper bound, namely

$$\min_K \min_{D \in \mathcal{D}} \| DND^{-1} \|_\infty$$

by alternating between minimizing $\| DN(K)D^{-1} \|_\infty$ with respect to either $K$ or $D$ (while holding the other fixed).
**DK-iteration**

The *DK*-iteration proceeds as follows:

1. **K-step**: Synthesize and $\mathcal{H}_\infty$ controller for the scaled problem,

   $$\min_K \| DN(K) D^{-1} \|_\infty \text{ with fixed } D(s)$$

2. **D-step**: Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed $N$.

3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.
Consider a two-input, two-output system with transfer function matrix

\[ G(s) = \begin{bmatrix} \frac{k_1}{T_1 s + 1} & -0.05 \\ 0.1 & \frac{0.1 s + 1}{k_2} \\ 0.3 s + 1 & \frac{T_2 s - 1}{k_2} \end{bmatrix} \]

where the coefficients \( k_1 \) and \( k_2 \) have nominal values 12 and 5, respectively, and relative uncertainty 15\%, and the time constants \( T_1 \) and \( T_2 \) have nominal values 0.2 and 0.7, respectively, and relative uncertainty 20\%.
The closed-loop system is described by

\[ z = Tzw, \quad z = \begin{bmatrix} z_S \\ z_K \end{bmatrix}, \quad w = \begin{bmatrix} r \\ d \end{bmatrix} \]

The performance weighting and control weighting functions are

\[ W_S(s) = \begin{bmatrix} w_S(s) & 0 \\ 0 & w_S(s) \end{bmatrix}, \quad W_K(s) = \begin{bmatrix} w_K(s) & 0 \\ 0 & w_K(s) \end{bmatrix}, \]

where

\[ w_S(s) = \frac{0.5}{s + 0.3}, \quad w_K(s) = \frac{0.001}{0.0001s + 1}. \]
Example

```matlab
clc; clf;
s = tf('s');
k1 = ureal('k1',12,'Percentage',15);
k2 = ureal('k2',5,'Percentage',15);
T1 = ureal('T1',0.2,'Percentage',20);
T2 = ureal('T2',0.7,'Percentage',20);
G = [ k1/(T1*s+1), -0.05/(0.1*s+1);
     0.1/(0.3*s+1), k2/(T2*s-1)];
ws = 0.5*(s+10)/(s+0.3);
wk = 0.1*(0.001*s+1)/(0.0001*s+1);
WS = [ws 0 ; 0 ws];
WK = [wk 0 ; 0 wk];

systemnames = ' G WS WK';
inputvar = '[r{2}; d{2}; u{2}]';
outputvar = '[WS; WK; r-G-d]; % e = r-G-d
input_to_G = '[ u ]';
input_to_WS = '[ r-G-d ]';
input_to_WK = '[ u ]';
sysIC = sysic;
nmeas = 2;
ncont = 2;
fv = logspace(-3,3,100);
opt = dkitopt('FrequencyVector',fv, ... 'DisplayWhileAutoIter','on', ... 'NumberOfAutoIterations',3)
[K,CL,BND,INFO] = dksyn(sysIC,nmeas,...
ncont,opt);
```
**DK-iteration**

**Example**

<table>
<thead>
<tr>
<th>Iteration Summary</th>
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<tbody>
<tr>
<td>------------------</td>
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<tr>
<td><strong>Iteration #</strong></td>
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<tr>
<td><strong>Controller Order</strong></td>
</tr>
<tr>
<td><strong>Total D-Scale Order</strong></td>
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<tr>
<td><strong>Gamma Achieved</strong></td>
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<tr>
<td><strong>Peak mu-Value</strong></td>
</tr>
</tbody>
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Lecture 12: Robust Stability and Robust Performance Analysis
DK-iteration

Example

Closed-loop robust performance

- \( \mu \)-upper bound
- \( \mu \)-lower bound

Lecture 12: Robust Stability and Robust Performance Analysis
Example

\[ \text{From in1 to out1} \]

\[ \text{From in2 to out1} \]

\[ \text{From in1 to out2} \]

\[ \text{From in2 to out2} \]
Reference

1. Herbert Werner "Lecture note on *Optimal and Robust Control*", 2012

2. Ryozo Nagamune ”Lecture note on *Multivariable Feedback Control*”, 2009

3. Sigurd Skogestad and Ian Postlethwaite, ”Multivariable Feedback Control”, 2008